**EE 313 Summer 2016**
**Midterm I**
**Prof. Hazem Hajj**

**1. a. Show that magnitude of complex exponential $|e^{j\theta t}|$ = 1 for any θ and any t.**

- $e^{j\theta t} = cos(\theta t) + jsin(\theta t)$ from Euler's identity.

$Magnitude = \sqrt{(real)^2 + (imaginary)^2} = \sqrt{(\cos(\theta t))^2 + (\sin(\theta t))^2} = 1$ since we know $cos^2(\alpha) + sin^2(\alpha) = 1$ for all α.

Thus, $|e^{j\theta t}|$ is 1 for any θ and any t.

**b. Consider the set of complex numbers z given by |z| < 3. Plot the set in the complex plane.**

- let z=a+jb

Since |z| < 3 we have $\sqrt{a^2 + b^2} < 3 \Rightarrow a^2 + b^2 < 9$

Thus, the region is within a circle of radius 3 in the complex plane.

**c. Given the complex number $z_1$ = -4+3j, find the polar representation forms for the roots $z_1^{1/4}$**

- we have, $z_1 = -4 + 3j$ $and$ $z_1^{\frac{1}{n}} = r^{1/n} e^{j\frac{(\theta + k\pi)}{n}}$ where k=0,1,2,… n-1

Here, $r = \sqrt{4^2 + 3^2} = 5$ $and$ $\theta = \tan^{-1}\left(\frac{3}{-4}\right) = 2.4981\ radians$

Thus,

$$z_1^{1/4} = 5^{1/4} e^{\frac{j(2.4981 + k\pi)}{4}}, k = 0,1,2,3$$

**d. Assume we have the complex numbers A and z, where they can be represented in polar form as A = a $e^{j\theta}$, and z = b $e^{j\beta}$ . Assume that their conjugates are represented by A* and z* respectively. Show that $\boldsymbol{Az^n + A^*(z^*)^n}$ cane be written as:**

$$\boldsymbol{Az^n + A^*(z^*)^n = x[n]cos(\beta n + \theta)}$$

**Also show the expression for x[n].**

- We have $A = ae^{j\vartheta}, A^* = ae^{-j\theta}, z = be^{j\beta}, z^* = be^{-j\beta}$

Thus,

$$Az^n + A^*(z^*)^n = ae^{j\theta}(be^{j\beta})^n + ae^{-j\theta}(be^{-j\beta})^n = ae^{j\theta}b^n e^{jn\beta} + ae^{-j\theta}b^n e^{-jn\beta}$$
$$= ab^n\left[e^{j(\theta + n\beta)} + e^{-j(\theta + n\beta)}\right] = 2ab^n \cos(\beta n + \theta)$$

And $x[n] = 2ab^n$

**2. Assume a continuous-time system is represented by the following differential equation $\boldsymbol{\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = 2\frac{dx}{dt} + x(t)}$, answer the following questions:**

**a. Find the characteristic roots for the system.**

- the characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$

The roots are -3 and -2

**b. Find the zero-input response for the system. Assume the initial conditions y(0) = 0, and y'(0) = 1.**

- The zero input response is :

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Using the given initial conditions get,

$$c_1 = -1 \; and \; c_2 = 1$$

Thus,

$$y_0(t) = -e^{-3t} + e^{-2t}$$

**c. Is the system asymptotically stable? Justify your answers.**

- since the characteristic roots obtained have negative real parts, thus both lie in left hand plane and thus it is asymptotically stable.

**d. Determine the impulse response of the system**

$$h(t) = b_0 \delta(t) + \big(P(D) y_n(t)\big) u(t)$$

Here, $b_0 = 0$, and since N=2, h(0)=0 and h'(0)=1

Thus,

$$h(t) = [5e^{-3t} - 3e^{-2t}] u(t)$$

**e. Is the system causal? Justify your answer.**

- Yes system is causal since y(t) depends on present values of x(t) and no future values. Also note that h(t) is zero for t<0.

**3. Assume a discrete-time system is represented by the following difference equation $y[n+2] + 5y[n+1] + 6y[n] = 2x[n+1] + x[n]$, answer the following questions with initial conditions y[-2] = 0, y[-1]=1.**

**a. Find the characteristic roots for the system.**

- the characteristic equation is $\gamma^2 + 5\gamma + 6 = 0$

The roots as -3 and -2

**b. Find the zero-input response.**

- The zero input response is:

$$y_0[n] = c_1(-3)^n + c_2(-2)^n$$

Using the given initial conditions get-

$$c_1 = -9 \; and \; c_2 = 4$$

$$y_0[n] = -9(-3)^n + 4(-2)^n$$

**c. Is the system asymptotically stable? Justify your answers.**

- the system is asymptotically unstable since the characteristic roots lies outside the unit circle in the complex plane.

**d. Find the impulse response.**

- For the given difference equation $a_N = 6 \; and \; b_N = 1$

Thus,

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_0[n]u[n]$$
$$h[n] = \frac{1}{6}\delta[n] + y_0[n]u[n]$$

$$h[n+2] + 5h[n+1] + 6h[n] = 2\delta[n+1] + \delta[n]$$

We know, h[-2]=h[-1]=0 and $\delta$[0]=1. Thus,
For n=-2,

$$h[0] + 5h[-1] + 6h[-2] = 2\delta[-1] + \delta[-2] => \ h[0] = 0$$

For n=-1,

$$h[1] + 5h[0] + 6h[-1] = 2\delta[0] + \delta[-1] \ => \ h[1] = 2$$

Thus,

$$h[n] = \frac{1}{6}\delta[n] + [c_1(-3)^n + c_2(-2)^n]u[n]$$

For n=0,

$$h[0] = \frac{1}{6}\delta[0] + [c_1 + c_2]u[0] => c_1 + c_2 = -1/6$$

For n=1,

$$h[1] = \frac{1}{6}\delta[1] + [c_1(-3) + c_2(-2)]u[1] => -3c_1 - 2c_2 = 0$$

$$c_1 = -1/3 \ and \ c_2 = 1/2$$

$$h[n] = \frac{1}{6}\delta[n] + [\left(-\frac{1}{3}\right)(-3)^n + \left(\frac{1}{2}\right)(-2)^n]u[n]$$

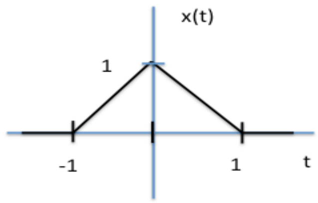
**4. a. Determine the zero-state output of a system represented by the impulse response $\boldsymbol{h(t)} = \frac{\boldsymbol{1}}{\boldsymbol{t+1}}\boldsymbol{u(t)}$, and the input to the system is the step function $\boldsymbol{x(t)} = \boldsymbol{4u(t)}$.**

- $y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t 4\,u(\tau)\frac{1}{t-\tau+1}u(t-\tau)d\tau = 4\int_0^t \frac{1}{t-\tau+1}d\tau$

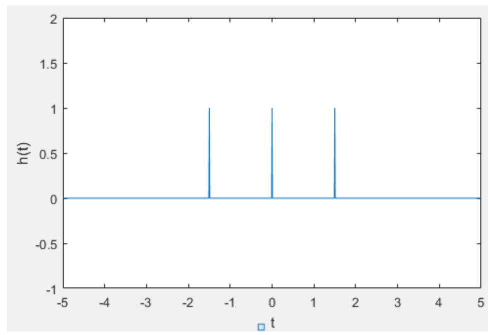
Let $t - \tau + 1 = s \Rightarrow \ -d\tau = ds$
Thus,

$$y(t) = 4\int_{t+1}^{1} -\frac{1}{s}ds = 4\int_{1}^{t+1}\frac{1}{s}ds = 4\ln(t+1)$$

**b. Assume the triangular wave shown below is fed to a system represented by the impulse response $\boldsymbol{h(t)} = \sum_{k=-1}^{k=1}\boldsymbol{\delta}(\boldsymbol{t} - \frac{3}{2}\boldsymbol{k})$, ($\boldsymbol{k}$ takes integer values), derive and plot the zero-state response.**




- $h(t) = \delta\left(t+\frac{3}{2}\right) + \delta(t) + \delta\left(t-\frac{3}{2}\right)$



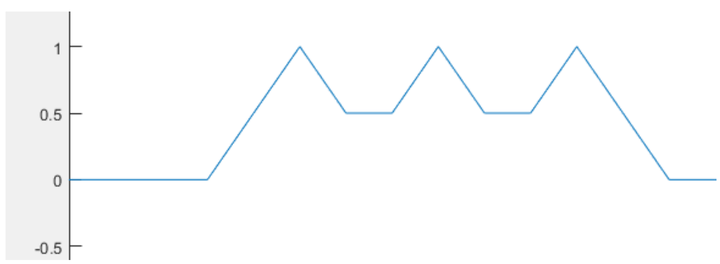

Thus,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau+3/2)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-3/2)d\tau$$

$$= x\left(t+\frac{3}{2}\right) + x(t) + x\left(t-\frac{3}{2}\right)$$

y(t) is of the form –

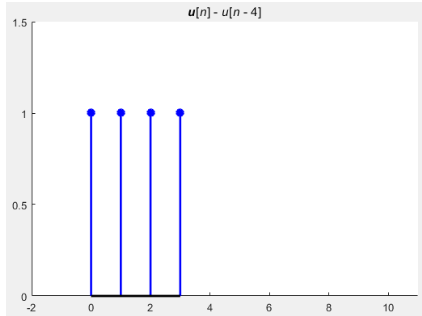



**5. a. Determine the zero-state output of a system represented by the impulse response $[\mathbf{n}] = (\mathbf{0.2})^{\boldsymbol{n}}\boldsymbol{u}[\boldsymbol{n}]$ , and the input to the system is the function $\mathbf{x}[\mathbf{n}] = (\mathbf{5})^{\boldsymbol{n}}\boldsymbol{u}[\boldsymbol{n}]$.**

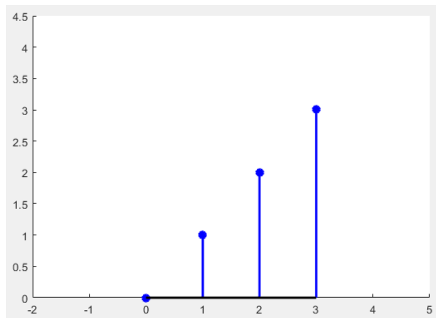
- $y[n] = \sum_0^n x[m]h[n-m] = \sum_0^n (0.2)^m u[m](5)^{n-m}u[n-m] = 5^n \sum_0^n \frac{0.2^m}{5} = \frac{(0.2)^{n+1}-5^{n+1}}{0.2-5}u[n]$

**b. Determine the zero-state output of a system represented by the impulse response $\boldsymbol{h}[\boldsymbol{n}] = \boldsymbol{n}(\boldsymbol{u}[\boldsymbol{n}] - \boldsymbol{u}[\boldsymbol{n}-\mathbf{4}])$ and when the input to the system is $\boldsymbol{x}[\boldsymbol{n}] = (\boldsymbol{u}[\boldsymbol{n}] - \boldsymbol{u}[\boldsymbol{n}-\mathbf{4}])$.**

- x[n] is:




h[n] is:




Proceeding with graphical convolution for n=0,1,2,3,4,5,6 –

y[0]=0

y[1]=1

y[2]=3

y[3]=6

y[4]=6

y[5]=5

y[6]=3

y[n] is 0 for n<0 and n>6